Kinetic Equations

Solution to the Exercises

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Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be a bounded vector field. Let $X \in C^1\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be the flow associated to b, i.e., the unique differentiable solution to

$$\begin{cases} \partial_s X(s,t,x) = b(s, X(s,t,x)), & \forall (s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d, \\ X(t,t,x) = x, & \forall (t,x) \in [0,T] \times \mathbb{R}^d \end{cases} \tag{1}$$

a Prove that X satisfies the semigroup property, i.e.,

$$X(r,s,X(s,t,x)) = X(r,t,x), \qquad \forall\, r,s,t \in [0,T], \ \forall x \in \mathbb{R}^d. \tag{2}$$

b Use point **a** to prove that for any $s,t \in [0,T]$ the map $X(s,t,\cdot) \in C^1\left(\mathbb{R}^d; \mathbb{R}^d\right)$ is a C^1 diffeomorphism, i.e. it is invertible with its inverse in $C^1\left(\mathbb{R}^d; \mathbb{R}^d\right)$.

Proof. For the proof of **a**, fix $s,t \in [0,L]$, $x \in \mathbb{R}^d$ and consider the functions $u(r) := X(r,s,X(s,t,x))$ and $v(r) := X(r,t,x)$. By definition of X, they both solve the problem

$$\begin{cases} \partial_r f(r) = b(r, f(r)), & r \in [0,T], \\ f(s) = X(s,t,x), \end{cases} \tag{3}$$

with s and x fixed. Given that the solution to this problem is unique, u is equal to v and this proves (2).

To prove **b**, using point **a** we get that for any $s,t \in [0,T]$, $x \in \mathbb{R}^d$ we get

$$X(t,s,X(s,t,x)) = x = X(s,t,X(t,s,x)). \tag{4}$$

This implies that $X(t,s,\cdot)$ is both a left and right inverse of $X(s,t,\cdot)$; given that $X(t,s,\cdot)$ is C^1, this proves **b**.

□

Exercise 2

Let $T \geqslant 0$ be a positive real number, $b \in C^1\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be a bounded vector field and $X \in C^1\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be again the flow associated to b. Define the *Jacobian* $J \in C\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}\right)$ as we did in class as

$$J(s,t,x) := \det(\nabla_x X)(s,t,x). \tag{5}$$

From classical results in the theory of ordinary differential equations, $\partial_s J$ exists and is in $C\left([0,T]\times[0,T]\times\mathbb{R}^d;\mathbb{R}\right)$.

Show that $J(s,t,x)>0$ for all $(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d$ and that J solves

$$\begin{cases} (\partial_s J)(s,t,x) = (\operatorname{div}_x b)(s,X(s,t,x))\,J(s,t,x)\,, & \forall (s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d, \\ J(t,t,x)=1, & \forall (t,x)\in[0,T]\times\mathbb{R}^d. \end{cases} \tag{6}$$

Prove moreover that J satisfies

$$\partial_t J(s,t,x) + \operatorname{div}_x\left(b(t,x)\,J(s,t,x)\right) = 0. \tag{7}$$

Hint: You can assume without proof that $\partial_s\nabla_x X$ exists and it is in $C\left([0,T]\times[0,T]\times\mathbb{R}^d;\mathbb{R}\right)$, and is equal to $\nabla_x\partial_s X$. For a proof of this result, one can look at Theorem 2.10 in the book ***Ordinary Differential Equations and Dynamical Systems*** *from Gerald Teschl, available online for free.*

Proof. From Exercise 1 we get that

$$X(s,t,X(t,s,x)) = x. \tag{8}$$

If we differentiate this family of equalities we get the equality of matrices given as:

$$(\nabla_x X)(t,s,x)\cdot(\nabla_x X)(s,t,X(t,s,x)) = \mathrm{id}_{\mathbb{R}^d}\,. \tag{9}$$

If we now do the determinant of both sides we get, by definition of Jacobian, that

$$J(t,s,x)\,J(s,t,X(t,s,x)) = 1. \tag{10}$$

This immediately implies that $J(s,t,x)\neq 0$ for any $(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d$; moreover, given that the sign of J is a continuous function, J itself is either going to be always positive or always negative. Given that $J(t,t,x)=\det(\nabla X)(t,t,x)=\det\mathrm{id}_{\mathbb{R}^d}=1$, we get that it needs to be always positive.

For the next part, first recall the *Jacobi's formula*, i.e., let $A\in C^1([0,T];GL_n(\mathbb{R}))$; then we have that $\det(A)\in C^1([0,T])$ and

$$\partial_t\det(A(t)) = (\det A(t))\operatorname{tr}\Big((A(t))^{-1}(\partial_t A)(t)\Big)\,. \tag{11}$$

As a consequence, the derivative of J can be computed as

$$\begin{aligned} \partial_s J(s,t,x) &= \partial_s\left(\det(\nabla_x X)(s,t,x)\right) && (12)\\ &= \det\left((\nabla_x X)(s,t,x)\right)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,\partial_s(\nabla_x X)(s,t,x)\right) && (13)\\ &= J(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,\partial_s(\nabla_x X)(s,t,x)\right). && (14) \end{aligned}$$

Recall that we can from the definition of X we get

$$\partial_s X(s,t,x) = b(s,X(s,t,x))\,. \tag{15}$$

Applying now the operator ∇_x we get

$$\partial_s \nabla_x X(s,t,x) = \nabla_x \partial_s X(s,t,x) = (\nabla_x X)(s,t,x)(\nabla_x b)(s, X(s,t,x)) . \tag{16}$$

Substituting back in the derivative of J and using (9) we get

$$\begin{aligned} \partial_s J(s,t,x) &= J(s,t,x) \operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))(\nabla_x X)(s,t,x)(\nabla_x b)(s,X(s,t,x))\right) &(17)\\ &= J(s,t,x)\operatorname{tr}\left((\nabla_x b)(s,X(s,t,x))\right). &(18)\end{aligned}$$

By definition of trace now we get that if F is a vector field, $\operatorname{tr}(\nabla_x F) = \operatorname{div}_x F$, and therefore

$$\partial_s J(s,t,x) = (\operatorname{div}_x b)(s, X(s,t,x)) J(s,t,x) . \tag{19}$$

Using the fact that $J(t,t,x) = \det(\nabla_x X)(t,t,x) = \det(\nabla_x x) = 1$ we get (6).

For the derivative in t of $J(s,t,x)$, we first recall that the derivative in t of $X(s,t,x)$ is given by

$$\partial_t X(s,t,x) = -b(t,x) \cdot (\nabla_x X)(s,t,x) . \tag{20}$$

As a consequence we get that

$$\begin{aligned} \partial_t J(s,t,x) &= \partial_t \left(\det(\nabla_x X)(s,t,x)\right) &(21)\\ &= \det(\nabla_x X)(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,\partial_t \nabla_x X(s,t,x)\right) &(22)\\ &= J(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,\nabla_x \partial_t X(s,t,x)\right). &(23)\end{aligned}$$

Applying the definition of the gradient we get

$$\begin{aligned} \nabla_x \partial_t X(s,t,x) &= -\nabla_x \left(b(t,x)\cdot(\nabla_x X)(s,t,x)\right) &(24)\\ &= -(\nabla_x b)(t,x)(\nabla_x X)(s,t,x) - b(t,x)\cdot \nabla_x(\nabla_x X)(s,t,x) &(25)\end{aligned}$$

On the other hand, we can consider what is the gradient in x of J; to do so, we consider the derivative along the j-th component of J to get

$$\begin{aligned} \partial_{x_j} J(s,t,x) &= \partial_{x_j} J(s,t,x) &(26)\\ &= \det(\nabla_x X)(s,t,x)\operatorname{tr}\big((\nabla_x X)(t,s,X(s,t,x))\,\partial_{x_j}\nabla_x X(s,t,x)\big) &(27)\\ &= J(s,t,x)\operatorname{tr}\big((\nabla_x X)(t,s,X(s,t,x))\,\partial_{x_j}\nabla_x X(s,t,x)\big). &(28)\end{aligned}$$

As a consequence we get

$$\begin{aligned} \partial_t J(s,t,x) &= J(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,\nabla_x\partial_t X(s,t,x)\right) &(29)\\ &= -J(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))(\nabla_x b)(t,x)(\nabla_x X)(s,t,x)\right) &(30)\\ &\quad - J(s,t,x)\operatorname{tr}\left((\nabla_x X)(t,s,X(s,t,x))\,b(t,x)\cdot\nabla_x(\nabla_x X)(s,t,x)\right). &(31)\end{aligned}$$

Using cyclicity of the trace and (9) we get for the term in (30) that

$$
\begin{aligned}
& - J(s,t,x)\operatorname{tr}\left(\left(\nabla_x X\right)(t,s,X(s,t,x))\left(\nabla_x b\right)(t,x)\left(\nabla_x X\right)(s,t,x)\right) = && (32)\\
& \quad = -J(s,t,x)\operatorname{tr}\left(\left(\nabla_x X\right)(s,t,x)\left(\nabla_x X\right)(t,s,X(s,t,x))\left(\nabla_x b\right)(t,x)\right) && (33)\\
& \quad = -J(s,t,x)\operatorname{tr}\left(\left(\nabla_x b\right)(t,x)\right) = -\operatorname{div}_x(b(t,x))\,J(s,t,x)\,. && (34)
\end{aligned}
$$

On the other hand, using the explicit form of the derivative along the j-th variable we get for (31)

$$
\begin{aligned}
& - J(s,t,x)\operatorname{tr}\left(\left(\nabla_x X\right)(t,s,X(s,t,x))\,b(t,x)\cdot\nabla_x\left(\nabla_x X\right)(s,t,x)\right) && (35)\\
& \quad = -J(s,t,x)\sum_{j=1}^{d} b_j(t,x)\operatorname{tr}\Big(\left(\nabla_x X\right)(t,s,X(s,t,x))\,\partial_{x_j}\left(\nabla_x X\right)(s,t,x)\Big) && (36)\\
& \quad = -\sum_{j=1}^{d} b_j(t,x)\,\partial_{x_j} J(s,t,x) = -b(t,x)\cdot\left(\nabla_x J\right)(s,t,x)\,. && (37)
\end{aligned}
$$

Summing all up we get that from basic properties of the divergence operator we get

$$
\begin{aligned}
\partial_t J(s,t,x) &= -\operatorname{div}_x(b(t,x))\,J(s,t,x) - b(t,x)\cdot\left(\nabla_x J\right)(s,t,x) && (38)\\
&= -\operatorname{div}\left(b(t,x,)\,J(s,t,x)\right), && (39)
\end{aligned}
$$

which gives us (7).

□

Exercise 3

Let $T \geqslant 0$ be a positive real number, $b \in C^2\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$ be a bounded vector-field. Assume that $u_0 \in C^1\left(\mathbb{R}^d\right)$ and that $f \in C^1\left([0,T]\times\mathbb{R}^d\right)$.

Prove that there exists a unique solution $u \in C^1\left([0,T]\times\mathbb{R}^d\right)$ for the inhomogeneous transport equation

$$
\begin{cases}
\partial_t u(t,x) + \operatorname{div}_x\left(b(t,x)\,u(t,x)\right) = f(t,x)\,, & \forall\,(t,x)\in[0,T]\times\mathbb{R}^d,\\
u(0,x) = u_0(x)\,, & \forall\; x\in\mathbb{R}^d.
\end{cases}
\tag{40}
$$

Proof. Consider first the flow X associated to b; given that $b \in C^2\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$, we get that $X \in C^2\left([0,T]\times[0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$. As above, define the Jacobian $J = \det\left(\nabla_x X\right) \in C^1\left([0,T]\times[0,T]\times\mathbb{R}^d\right)$.

To prove existence of a solution, consider the following function

$$
v(t,x) := u_0(X(0,t,x))\,J(0,t,x) + \int_0^t ds\; f(s,X(s,t,x))\,J(s,t,x)\,. \tag{41}
$$

If we define $F(s,t,x) := f(s,X(s,t,x))\,J(s,t,x)$ and we use (7), we can get

$$\begin{aligned}
\partial_t F(s,t,x) &= (\partial_t X)(s,t,x)\cdot(\nabla_x f)(s,X(s,t,x))\,J(s,t,x) && (42)\\
&\quad - f(s,X(s,t,x))\,\mathrm{div}_x\left(b(t,x)\,J(s,t,x)\right) && (43)\\
&= -\left(b(t,x)\cdot\nabla_x X\right)(s,t,x)\cdot(\nabla_x f)(s,X(s,t,x))\,J(s,t,x) && (44)\\
&\quad - f(s,X(s,t,x))\,\mathrm{div}_x\left(b(t,x)\,J(s,t,x)\right) && (45)\\
&= -b(t,x)\cdot\nabla_x\left(f(s,X(s,t,x))\right)J(s,t,x) && (46)\\
&\quad - f(s,X(s,t,x))\,\mathrm{div}_x\left(b(t,x)\,J(s,t,x)\right) && (47)\\
&= -\,\mathrm{div}_x\left(b(t,x)\,f(s,X(s,t,x))\,J(s,t,x)\right) && (48)\\
&= -\,\mathrm{div}_x\left(b(t,x)\,F(s,t,x)\right). && (49)
\end{aligned}$$

On the other hand, using (7) again we get

$$\begin{aligned}
&\partial_t\left(u_0(X(s,t,x))\,J(0,t,x)\right) = && (50)\\
&= (\partial_t X)(s,t,x)\cdot(\nabla_x u_0)(X(s,t,x))\,J(0,t,x) && (51)\\
&\quad + u_0(X(s,t,x))\,\partial_t J(0,t,x) && (52)\\
&= -\left((b(t,x))\cdot\nabla_x X\right)(s,t,x)\cdot(\nabla_x u_0)(X(s,t,x))\,J(0,t,x) && (53)\\
&\quad - u_0(X(s,t,x))\,\mathrm{div}_x\left(b(t,x)\,J(0,t,x)\right) && (54)\\
&= -\left(b(t,x)\right)\cdot\nabla_x\left(u_0(X(s,t,x))\right)J(0,t,x) && (55)\\
&\quad - u_0(X(s,t,x))\,\mathrm{div}_x\left(b(t,x)\,J(0,t,x)\right) && (56)\\
&= -\,\mathrm{div}_x\left(b(t,x)\,u_0(X(s,t,x))\,J(0,t,x)\right). && (57)
\end{aligned}$$

Collecting the previous estimates and using the fundamental theorem of calculus, we get

$$\begin{aligned}
\partial_t v(t,x) &= \partial_t\left(u_0(X(s,t,x))\,J(0,t,x)\right) + F(t,t,x) + \int_0^t ds\ \partial_t F(s,t,x) && (58)\\
&= -\,\mathrm{div}_x\left(b(t,x)\,u_0(X(s,t,x))\,J(0,t,x)\right) + f(t,x) && (59)\\
&\quad - \int_0^t ds\ \ \mathrm{div}_x\left(b(t,x)\,F(s,t,x)\right) && (60)\\
&= f(t,x) - \mathrm{div}_x\left(b(t,x)\,v(t,x)\right), && (61)
\end{aligned}$$

and therefore v is solution to (40).

Consider now u a solution to (40). From the definition of X and (6) we get

$$\begin{aligned}
\partial_s\left(u(s,X(s,t,x))\,J(s,t,x)\right) &= (\partial_t u)(s,X(s,t,x))\,J(s,t,x) + && (62)\\
&\quad + (\partial_s X)(s,t,x)\cdot(\nabla_x)(s,X(s,t,x))\,J(s,t,x) && (63)\\
&\quad + u(s,X(s,t,x))\,(\partial_s J)(s,t,x) && (64)\\
&= f(s,X(s,t,x))\,J(s,t,x). && (65)
\end{aligned}$$

If we integrate in s we now get

$$\begin{aligned}
\int_0^t ds\ f(s,X(s,t,x))\,J(s,t,x) &= \left[u(s,X(s,t,x))\,J(s,t,x)\right]_{s=0}^{s=t} && (66)\\
&= u(t,x) - u_0(X(0,t,x))\,J(0,t,x). && (67)
\end{aligned}$$

This means that if u is a solution, u must be equal to v defined above, and the solution is therefore unique.

□